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A class of initial value problems for 2×2 hyperbolic systems with relaxation

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ABSTRACT

The global existence and uniform BV estimates of weak solutions to a class of initial value problems of general 2×2 genuinely nonlinear strictly hyperbolic conservation laws with relaxation are proved, which also yield the convergence to weak solutions of the equilibrium equation as the relaxation parameter tends to zero. The results obtained in this paper extend the results in Luo and Yang (2004) [17] for a flood wave equation to the general 2×2 system. Also, the constraint as required in Luo and Yang (2004) [17] on the end states at $x = \pm\infty$, which says that the end states at $x = \pm\infty$ are on the equilibrium, is removed.

1. Introduction

The purpose of this paper is to consider a class of initial value problems for the following general 2×2 genuinely nonlinear strictly hyperbolic conservation laws with relaxation

$$u_t + f(u, v)_x = 0, \qquad v_t + g(u, v)_x = \frac{\Phi(u, v)}{\epsilon}, \tag{1.1}$$

where u and v are real functions of time variable $t \geqslant 0$ and spatial variable $x \in \mathbb{R}^1$, f, g and Φ are real functions of u and v, $\epsilon > 0$ is the small relaxation parameter. We will seek global week solutions to (1.1) with the initial data

$$\big(u(x, 0), v(x, 0)\big) = \big(u_0(x), v_0(x)\big), \quad x \in \mathbb{R}^1, \tag{1.2}$$

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for some bounded and measurable functions $u_0(x)$ and $v_0(x)$. The relaxation term is assumed to satisfy

$$\Phi_v(u,v) < 0, \qquad \Phi\big(u, v_*(u)\big) = 0, \tag{1.3}$$

for all (u,v) under consideration. Here and thereafter $A_u(u,v)$ and $A_v(u,v)$ are denoted as the partial derivatives of the function $A(u,v)$ with respect to the first and second independent variables. When the relaxation term $\Phi(u,v)$ vanishes, the system is in equilibrium and the equilibrium equation corresponding to (1.1) is given by

$$u_t + f_*(u)_x = 0, \qquad v = v_*(u), \tag{1.4}$$

where $f_*(u) = f(u, v_*(u))$. We will study the global existence of weak solutions for the problem (1.1) and (1.2) by assuming the following sub-characteristic condition (cf. [14])

$$\lambda_1(u,v) < f_*'(u) < \lambda_2(u,v), \tag{1.5}$$

for all (u,v) under consideration, where λ_1 and λ_2 are two distinct characteristic wave speeds. The initial data considered in this paper is assumed to satisfy a certain order condition as in [20], which states that certain initial data gives rise to a specific form of Riemann problem. More specifically, if we let

$$(u_i, v_i) = \big(u_0(x_i), v_0(x_i)\big), \quad i = 1, 2, \ \text{for } x_1 < x_2,$$

then we assume that the Riemann problem for the corresponding homogeneous system

$$u_t + f(u,v)_x = 0, \qquad v_t + g(u,v)_x = 0 \tag{1.6}$$

with the initial data

$$(u,v)(x,0) = (u_1, v_1), \quad \text{as } x < 0, \qquad (u,v)(x,0) = (u_2, v_2), \quad \text{as } x > 0,$$

is solvable by a shock wave of one characteristic family and a rarefaction wave of another family. We also assume that the system (1.6) satisfies the Glimm–Lax shock interaction condition (cf. [8]) which states that the interaction of two shocks of the same characteristic produces a shock of this family and a rarefaction wave of another family. In addition, as in [20], we require that all shock waves under consideration arise from the intersection of two characteristic families from exactly one characteristic family (see condition $(\mathrm{L})_i$ in Lemma 2.2). Under those assumptions, Smoller and Johnson (cf. [20]) obtained the global existence of the BV solutions to the homogeneous system (1.6) for the initial data satisfying the order condition mentioned above, which generalizes Zhang and Guo's results (cf. [24]) for the p-system of isentropic gas dynamics in the Lagrangian coordinates. One of the main purposes of this paper is to extend Smoller and Johnson's results in [20] to the relaxation system (1.1) to prove the global existence of BV solutions and obtain the uniform total variation estimates both in time and the relaxation parameter ϵ so that one can pass to the zero relaxation limit. For a flood wave model in [22], which is a p-system in the Lagrangian coordinates, the global BV solutions and uniform total variation estimates were obtained in [17] for the initial data satisfying the above mentioned order condition. The results obtained in the present paper extend the results in [17] for the flood wave equations to the general 2×2 systems. Also, we remove the constraint as required in [17] on the end states at $x = \pm\infty$, which says that the end states at $x = \pm\infty$ are on the equilibrium.

The method employed to construct the global weak solutions is a modified Glimm scheme used in [5] for general $n \times n$ inhomogeneous hyperbolic systems of conservation laws. In the study of global existence of weak solutions for hyperbolic conservation laws in one space dimension, the first global existence result of weak solution for the initial value problems for the general homogeneous strictly hyperbolic conservation laws was due to Glimm by using the celebrated random choice method (cf. [7]) for the initial data having small total variation. As far as the initial data with large total variation is concerned, Nishida (cf. [18]) proved the global existence of global BV solutions for the p-system of isothermal gas dynamics in the Lagrangian coordinates for $p(v) = 1/v$ by utilizing the special geometry of the shock curves in the Riemann invariants plane. For hyperbolic systems with relaxation, the uniform BV estimates are important to the understanding of the qualitative behavior of solutions such as wave interactions and zero relaxation limit. For a semi-linear constant coefficients model, the Jin–Xin model (cf. [9]), the uniform BV estimates and the zero relaxation limits were studied in [2]. For the global existence and uniform BV estimates of solutions to the quasilinear hyperbolic systems with relaxation, to the best knowledge of the author, the only available results are for some special systems. For example, the global existence of BV solution for the p-system with relaxation was obtained in [16] and [1], respectively, for the special pressure function $p(v) = 1/v$. Again, the special geometry of shock curves for this system plays a very important role in their analysis. The global existence of BV solutions of p-system with relaxation was established in [4] for general function p. However, the total variation bounds obtained in [4] depend on the relaxation parameter which do not allow passing the zero relaxation limit. Some other uniform BV estimates results were proved only for systems including a simple model in viscoelasticity considered in [15,23,19], which is in the class of Temple (cf. [21]) and some traffic flow models (cf. [12,13]).

For the general 2×2 relaxation system of conservation laws, the zero relaxation limit was studied in [10] by assuming that there is a uniform L^∞-bound for the solutions of the relaxation systems by the method of compensated compactness.

For the homogeneous system (1.6), the local behavior and large time behavior of solutions are all represented by the Riemann solutions of (1.6) (cf. [8]). This fact is important to deriving the global-in-time total variation estimates by using the Glimm scheme. However, the local behavior of solutions to the relaxation system (1.1) is similar to those of the homogeneous system (1.6). However, the large time behavior of solutions to (1.1) is represented by those of the equilibrium equation (1.4) (cf. [14]). This makes it challenging to derive the uniform total variation estimates for the relaxation systems. Global BV solutions for a class of the inhomogeneous systems for which the inhomogeneous terms are completely dissipative were established in [5]. However, the relaxation is only partially dissipative, and the results in [5] do not apply. For the p-system with frictional damping, the global existence of BV solutions was obtained in [3] and [6] by redistributing the damping and using the arguments in [5] and an entropy estimates. For the relaxation system (1.1), it seems very difficult to adopt this approach.

The rest of this paper is organized as follows. In Section 2, we will recall some results in [20] and state the main theorems of this paper. Section 3 is devoted to the proofs of the theorems stated in Section 2.

2. Preliminaries and main results

In this section, we first review some results in [20] for the 2×2 system of hyperbolic conservation laws (1.6). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by

$$F : (u, v) \mapsto \big(f(u, v), g(u, v)\big),$$

and denote the Jacobian matrix of F by $dF(u, v)$. We assume that, as in [20],

$$f_v g_u > 0 \quad \text{in an open set } U \subseteq \mathbb{R}^2.$$

This would imply that the system is strictly hyperbolic. For definiteness we will assume that

$$f_v < 0 \quad \text{and} \quad g_u < 0 \quad \text{in } U. \tag{2.1}$$

Let $\lambda_1(u,v) < \lambda_2(u,v)$ be two real eigenvalues of $dF(u,v)$ for $(u,v) \in U$. Denote by $l_i(u,v)$ and $r_i(u,v)$, $i = 1,2$, the corresponding left and right eigenvectors which are in the form

$$r_1 = (1, a_1)^t, \qquad r_2 = (-1, -a_2)^t, \qquad l_1 = (-a_2, 1), \qquad l_2 = (-a_1, 1),$$

where $a_i = g_u/(\lambda_i - g_v)$, $i = 1,2$.

We assume that the hyperbolic system (1.1) is genuinely nonlinear in U (cf. [11]),

$$\nabla\lambda_i(u,v) \cdot r_i(u,v) > 0, \quad (u,v) \in U,\ i = 1,2,$$

or equivalently,

$$l_i(u,v) d^2 F\big(r_i(u,v), r_i(u,v)\big) > 0, \quad (u,v) \in U,\ i = 1,2. \tag{2.2}$$

We will also assume that

$$l_j(u,v) d^2 F\big(r_i(u,v), r_i(u,v)\big) > 0, \quad (u,v) \in U,\ i,j = 1,2,\ i \neq j. \tag{2.3}$$

It was shown in [20] that the interaction of two shocks of the same characteristic family produces a shock of that characteristic family plus a rarefaction wave of another characteristic family as (2.3) holds. We may write (2.2) and (2.3) in the form

$$l_j(u,v) d^2 F\big(r_i(u,v), r_i(u,v)\big) > 0, \quad (u,v) \in U,\ i,j = 1,2. \tag{2.4}$$

Smoller and Johnson (cf. [20]) constructed wave curves $v = w(u; P_0)$ and $v = s(u; P_0)$ which represent states that can be connected to P_0 on the right by rarefaction waves and shock waves, respectively. It is well known that there exists a transformation of independent variables $(u,v) \to (\omega, z)$ satisfying

$$m(u,v) \equiv z_v(u,v) > 0, \qquad n(u,v) \equiv \omega_v(u,v) > 0, \tag{2.5}$$

and

$$z_u(u,v) = -a_1(u,v) m(u,v), \qquad \omega_u(u,v) = -a_2(u,v) n(u,v), \tag{2.6}$$

for all (u,v) under consideration, which puts the system (1.6) into Riemann invariant form:

$$\omega_t + \lambda_1 \omega_x = 0, \qquad z_t + \lambda_2 z_x = 0,$$

and (ω, z) is called the Riemann invariants. A 1 (respectively 2)-rarefaction wave is an open region in the x–t plane in which z (respectively ω) is constant. Since we will only be concerned with rarefaction waves from a single family, we will just consider 1-rarefaction waves. Since l_2 is parallel to (z_u, z_v), so that r_1 is perpendicular to (z_u, z_v). Hence the curve $z(u,v) = \text{constant}$ satisfies

$$\frac{dv}{du} = a_1. \tag{2.7}$$

Let the solution of (2.7) through P_0 be denoted by $v = w(u; P_0)$. The shock curves satisfy the Rankine–Hugoniot conditions

$$\sigma(u-u_0) = f(u,v) - f(u_0,v_0), \qquad \sigma(v-v_0) = g(u,v) - g(u_0,v_0),$$

where $\sigma = \sigma(u,v;u_0,v_0)$ is the shock speed. This determines two curves $v = s_1(u;P_0)$ and $v = s_2(u;P_0)$ on the u–v plane. We define an i-shock, $i = 1,2$, to be a discontinuity $x = x_i(t)$ of the function $(u(x,t), v(x,t))$ satisfying the Rankine–Hugoniot condition and the single equality

$$\lambda_i\big(u(x+0,t), v(x+0,t)\big) < x_i'(t) < \lambda_i\big(u(x-0,t), v(x-0,t)\big),$$

where $x_i'(t) = \sigma(u(x+0,t), v(x+0,t); u(x-0,t), v(x-0,t))$ is the shock speed. With these definitions, we state the properties of these wave curves in the subsequent lemmas.

Lemma 2.1. *(See Smoller and Johnson [20].) Let the system* (1.6) *satisfy* (2.1) *and* (2.4) *in an open set* $U \subseteq \mathbb{R}^2$*. Then through any point* $P_0 = (u_0, v_0) \in U$*,*

(1) *there exists an increasing* (*respectively decreasing*)*, convex downward* (*respectively upward*) *curve* $v = w(u;P_0)$ *defined for all* $u \geqslant u_0$ (*respectively* $u \leqslant u_0$) *for which* $(u, w(u;P_0)) \in U$ *such that any point on this curve is a state which can be connected to* P_0 *on the right by a* 1 (*respectively* 2)*-rarefaction wave*;
(2) *there exists a decreasing* (*respectively increasing*)*, convex upward* (*respectively downward*) *curve* $v = s(u;P_0)$ *defined for all* $u \geqslant u_0$ (*respectively* $u \leqslant u_0$) *for which* $(u, s(u;P_0)) \in U$ *such that any point on this curve is a state which can be connected to* P_0 *on the right by a* 2 (*respectively* 1)*-shock wave.*

For the homogeneous system (1.6), there is an ordering principle due to Smoller and Johnson in [20] (this ordering principle was first found by Zhang and Guo (cf. [24]) for the p-system of isentropic gas dynamics in Lagrangian coordinates). Let $P_0 = (u_0, v_0) \in U$ and

$$C_1(P_0) \equiv \big\{(u,v) \in U\colon\ s_1(u;P_0) \leqslant v \leqslant w_2(u;P_0),\ u \leqslant u_0\big\},$$
$$C_2(P_0) \equiv \big\{(u,v) \in U\colon\ s_2(u;P_0) \leqslant v \leqslant w_1(u;P_0),\ u \geqslant u_0\big\},$$

where $s_i(u;P_0)$ and $w_i(u;P_0)$ denote i-shock wave and i-rarefaction wave, respectively.

Lemma 2.2. *(See Smoller and Johnson [20].) Suppose that the system* (1.6) *satisfies* (2.1) *and* (2.4) *in* U*. Assume that condition* $(\mathrm{L})_i$ ($i = 1$ *or* $i = 2$) *holds, then*

$$C_i(P_1) \subseteq C_i(P_0), \quad \textit{for every } P_1 \in C_i(P_0),\ P_0, P_1 \in U.$$

Here the conditions $(\mathrm{L})_i$ *are*

$$(\mathrm{L})_1 \quad \textit{for each } P_0 \in U,\ \textit{if } \big(u, s_1(u;P_0)\big) \in U,\ \textit{then } \sigma\big(u, s_1(u;P_0); P_0\big) < \lambda_2(P_0),$$
$$(\mathrm{L})_2 \quad \textit{for each } P_0 \in U,\ \textit{if } \big(u, s_2(u;P_0)\big) \in U,\ \textit{then } \sigma\big(u, s_2(u;P_0); P_0\big) > \lambda_1(P_0),$$

where $\sigma(u,v;P_0)$ *is the shock speed.*

Remark 2.3. The condition $(\mathrm{L})_i$ $(i = 1,2)$ assures that all i-shocks under consideration, arise from the intersection of characteristics from ith characteristic family, and not from the other family. Lemma 2.5 in [20] gives some sufficient conditions for $(L)_i$ to hold. In fact, $(\mathrm{L})_i$ is a part of the definition of Lax's shock (cf. [11]).

We now turn to stating our main results. For the bounded measurable initial data (u_0, v_0), we assume that

$$\text{(A1)} \quad \text{for any } x_1 < x_2,\ \big(u_0(x_2), v_0(x_2)\big) \in C_1\big(u_0(x_1), v_0(x_1)\big),$$

or

$$\text{(A2)} \quad \text{for any } x_1 < x_2,\ \big(u_0(x_2), v_0(x_2)\big) \in C_2\big(u_0(x_1), v_0(x_1)\big).$$

Therefore, $u_0(x)$ is monotone and $v_0(x)$ has bounded total variation, so we can set

$$\big(u^{\pm}, v^{\pm}\big) = \lim_{x\to\pm\infty}\big(u_0(x), v_0(x)\big).$$

With the preparation of notations, we can state our main result now.

Theorem 2.4. *Suppose that* (1.3) *and* (1.5) *and the assumptions in Lemma* 2.2 *are satisfied. Assume that the initial data satisfies* (A1) *or* (A2)*. Then there exists a positive constant δ such that if*

$$\big|u^+ - u^-\big| + \big|v^- - v_*\big(u^-\big)\big| \leqslant \delta, \tag{2.8}$$

the Cauchy problem (1.1) *and* (1.2) *admits a weak solution $(u^\epsilon, v^\epsilon)(x,t)$ for all $t \geqslant 0$, having the following properties*: *for all $x_1 \leqslant x_2$ and $t \geqslant 0$,*

$$u^- \geqslant u^\epsilon(x_1,t) \geqslant u^\epsilon(x_2,t) \geqslant u^+, \quad \text{and} \quad TV\, v^\epsilon(\cdot,t) \leqslant K\big|u^+ - u^-\big|, \tag{2.9}$$

if (A1) *holds, or*

$$u^- \leqslant u^\epsilon(x_1,t) \leqslant u^\epsilon(x_2,t) \leqslant u^+, \quad \text{and} \quad TV\, v^\epsilon(\cdot,t) \leqslant K\big|u^+ - u^-\big|, \tag{2.10}$$

if (A2) *holds, where K is a constant independent of t and ϵ. Moreover, there exist constants C, c and $\nu \in (0,1)$, independent of ϵ, such that*

$$\int_{-\infty}^{\infty} \big|u^\epsilon(x,t_2) - u^\epsilon(x,t_1)\big|\,dx \leqslant C|t_2 - t_1|, \tag{2.11}$$

$$\int_{-L}^{L} \big|v^\epsilon(x,t) - v_*\big(u^\epsilon(x,t)\big)\big|\,dx \leqslant C\big[\epsilon + L\nu^{ct/\epsilon}\big], \tag{2.12}$$

for any $t_1, t_2, t, L \geqslant 0$; and for any $h > 0$, $L \geqslant 0$, and $t_1, t_2 \geqslant h$,

$$\int_{-L}^{L} \big|v^\epsilon(x,t_2) - v^\epsilon(x,t_1)\big|\,dx \leqslant C|t_2 - t_1|\big(1 + L\epsilon^{-1}\nu^{ch/\epsilon}\big). \tag{2.13}$$

Furthermore, $(u^\epsilon, v^\epsilon)(x,t)$ satisfies the entropy condition

$$\partial_t\eta\big(u^\epsilon, v^\epsilon\big) + \partial_x q\big(u^\epsilon, v^\epsilon\big) - \frac{1}{\epsilon}\eta_v\Phi\big(u^\epsilon, v^\epsilon\big) \leqslant 0, \tag{2.14}$$

in the sense of distribution for any convex entropy flux pair (η, q) with $\nabla q(u,v) = \nabla\eta(u,v)dF(u,v)$.

Due to the uniform estimates (2.9) (or (2.10)), (2.11), (2.13) and (2.12), we can pass to the zero relaxation limit to obtain the following theorem:

Theorem 2.5. *Let $(u^\varepsilon, v^\varepsilon)(x,t)$ be the sequence of solutions to the Cauchy problem* (1.1) *and* (1.2) *given by Theorem* 2.4*. Then there exists a subsequence $\{(u^\varepsilon, v^\varepsilon)\}(x,t)$ which converges almost everywhere to a pair of functions $(u,v)(x,t)$ as $\varepsilon \to 0$. Moreover, the limit functions satisfy*

$$v(x,t) = v_*\big(u(x,t)\big) \quad \textit{for } t > 0 \textit{ a.e.},$$

and u is a weak solution to the Cauchy problem (1.4) *with initial data $u_0(x)$.*

In the following, we will only prove Theorems 2.4 and 2.5 for the case when (A2) holds. The case when (A1) holds is completely similar.

3. The modified Glimm scheme and proofs for Theorems 2.4 and 2.5

We use a modified Glimm scheme as in [5]. At first, we have to select a space mesh-length r and a time mesh-length s satisfying the CFL condition. In order to describe the CFL condition here, we need to know the maximal wave speed for the solutions. So we want to have a priori L^∞-bound for the solutions. First, we fix a pair of Riemann invariants (ω, z) defined in (2.5)–(2.6) and denote

$$B = \text{convex hull of } \left\{ \begin{array}{l} (u,v)\colon\ u^- \leqslant u \leqslant u^+, \\ \quad z\big(u^+, \min\big\{v^+, v_*\big(u^+\big)\big\}\big) \leqslant z(u,v) \leqslant z\big(u^-, \max\big\{v^-, v_*\big(u^-\big)\big\}\big) \end{array} \right\}.$$

Here the choice of the set B is based on the assumption (A2), meaning that the solutions will be in the region bounded by $u = u^-$, $u = u^+$, and the rarefaction waves $v = w(u; (u^-, \max\{v^-, v_*(u^-)\}))$ and $v = w(u; (u^+, \min\{v^+, v_*(u^+)\}))$. If (A1) is satisfied, one can choose the corresponding set. For any fixed $\epsilon > 0$, we choose time mesh-length s such that

$$1 - (s/\epsilon) \max_{(u,v)\in B} \big|\Phi_v(u,v)\big| > 0. \tag{3.1}$$

The CFL condition now is given by

$$r/s > \max_{(u,v)\in B} \max\big\{\big|\lambda_1(u,v)\big|, \big|\lambda_2(u,v)\big|\big\}.$$

Let $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ be an equidistributed sequence of random number in $(-1,1)$. After partitioning the upper half of the (x,t) plane into strips

$$T_n = \big\{(x,t)\colon -\infty < x < +\infty,\ ns \leqslant t < (n+1)s\big\}, \quad n = 0,1,2,\ldots,$$

we initiate the construction of the approximate solutions (u^s, v^s) by letting

$$\big(u^s(x,0-), v^s(x,0-)\big) = \big(u_0(x), v_0(x)\big). \tag{3.2}$$

Assuming that (u^s, v^s) has already been determined on $\bigcup_{j=0}^{n-1} T_j$, we extend (u^s, v^s) to T_n as the admissible solution of the Cauchy problem

$$u_t + f(u,v)_x = 0, \qquad v_t + g(u,v)_x = 0, \tag{3.3}$$

with the initial data at $t = ns$ as

$$u^s(x, ns) = u^s\big((m+\alpha_n)r, ns-\big), \qquad v^s(x, ns) = \left(v^s + \frac{s}{\epsilon}\Phi\big(u^s, v^s\big)\right)\big((m+\alpha_n)r, ns-\big),$$

for $(m-1)r < x < (m+1)r$, $m+n$ odd.

In order to prove Theorem 2.4, we need the following theorem for the sequence (u^s, v^s):

Theorem 3.1. *Suppose that* (1.3) *and* (1.5) *and the assumptions in Lemma* 2.2 *are satisfied. Assume that the initial data satisfies* (A2)*. Then there exists a positive constant δ such that if*

$$\big|u^+ - u^-\big| + \big|v^- - v_*\big(u^-\big)\big| \leqslant \delta,$$

the sequence $\{(u^s, v^s)(x,t)\}$ is well defined for all $t > 0$ and has the following properties: for any $x_1 < x_2$ and $t > 0$,

$$\big(u^s(x_2, t), v^s(x_2, t)\big) \in C_2\big(\big(u^s(x_1, t), v^s(x_1, t)\big)\big), \tag{3.4}$$

and

$$TV\,u^s(\cdot, t) + TV\,v^s(\cdot, t) \leqslant K\big|u^+ - u^-\big|, \tag{3.5}$$

where K is a constant independent of t and ϵ; and for any $x \in \mathbb{R}$ and $t > 0$,

$$\big(u^s, v^s\big)(x, t) \in B. \tag{3.6}$$

Moreover, there exist constants C, c and $\nu \in (0, 1)$, independent of ϵ, s, r and θ, such that

$$\int_{-\infty}^{\infty} \big|u^s(x, t_2) - u^s(x, t_1)\big|\,dx \leqslant C\big(|t_2 - t_1| + s\big), \tag{3.7}$$

$$\int_{-L}^{L} \big|v^s(x, t_2) - v^s(x, t_1)\big|\,dx \leqslant C(1 + L/\epsilon)\big(|t_2 - t_1| + s\big), \tag{3.8}$$

$$\int_{-L}^{L} \big|v^s(x, t) - v_*\big(u^s(x, t)\big)\big|\,dx \leqslant C\big[\epsilon + L\nu^{ct/\epsilon}\big], \tag{3.9}$$

for any $t_1, t_2, t, L \geqslant 0$.

This theorem can be proved by using Lemmas 2.2 and 3.2, which state that the ordering principle still holds when we consider the relaxation effect in the scheme provided the sub-characteristic condition is satisfied.

Lemma 3.2 *(Relaxation effect lemma). Suppose that* (1.3) *and* (1.5) *and the assumptions in Lemma* 2.2 *are satisfied. Then there exist positive constants δ_1, δ_2 and δ_3 such that if $(u_r, v_r) \in C_2((u_l, v_l))$, then $(u_r, \bar{v}_r) \in C_2((u_l, \bar{v}_l))$, provided that*

$$\big|v_r - v_*(u_r)\big| + \big|v_l - v_*(u_l)\big| \leqslant \delta_1, \qquad |u_r - u_l| + |v_r - v_l| \leqslant \delta_2, \quad \textit{and} \quad 0 < s/\epsilon \leqslant \delta_3. \tag{3.10}$$

Here we use the notation $\bar{v}_l = v_l + (s/\epsilon)\Phi(u_l, v_l)$ and $\bar{v}_r = v_r + (s/\epsilon)\Phi(u_r, v_r)$.

3.1. Proof of Lemma 3.2

We show that

$$s_2\big(u_r; (u_l, \bar{v}_l)\big) \leqslant \bar{v}_r \leqslant w_1\big(u_r; (u_l, \bar{v}_l)\big), \tag{3.11}$$

under the assumption $(u_r, v_r) \in C_2((u_l, v_l))$, i.e.,

$$s_2\big(u_r; (u_l, v_l)\big) \leqslant v_r \leqslant w_1\big(u_r; (u_l, v_l)\big), \qquad \Delta u \equiv u_r - u_l \geqslant 0. \tag{3.12}$$

(For the simplicity of the notation, here and in the following we use ΔA to denote $A_r - A_l$ for a quantity A.) (3.11) is proved in the following steps.

Step 1. In this step, we show that $\bar{v}_r \leqslant w_1(u_r; (u_l, \bar{v}_l))$, which is equivalent to

$$\Delta\bar{z} \equiv z(u_r, \bar{v}_r) - z(u_l, \bar{v}_l) \leqslant 0.$$

In view of (3.12), and the fact that

$$\begin{aligned}
\Delta\bar{z} &= \big[z(u_r, \bar{v}_r) - z(u_r, v_r)\big] + \big[z(u_r, v_r) - z(u_l, v_l)\big] + \big[z(u_l, v_l) - z(u_l, \bar{v}_l)\big] \\
&= \int_0^{\bar{v}_r - v_r} m(u_r, v + v_r)\, dv + \Delta z + \int_{\bar{v}_l - v_l}^{0} m(u_l, v + v_l)\, dv \\
&= \Delta z + \underbrace{\int_0^{\bar{v}_r - v_r} \big[m(u_r, v + v_r) - m(u_l, v + v_l)\big]\, dv}_{I_1} + \underbrace{\int_{\bar{v}_l - v_l}^{\bar{v}_r - v_r} m(u_l, v + v_l)\, dv}_{I_2},
\end{aligned}$$

we have $\Delta z \equiv z(u_r, v_r) - z(u_l, v_l) \leqslant 0$, which implies that

$$\Delta\bar{z} = \Delta z + I_1 + I_2. \tag{3.13}$$

The terms I_1 and I_2 can be handled as follows. First, we can write

$$I_1 = (\bar{v}_r - v_r)\big[m(u_r, v_\xi + v_r) - m(u_l, v_\xi + v_l)\big]$$

for some v_ξ between 0 and $\bar{v}_r - v_r$, which yields that

$$|I_1| \leqslant q_1(s/\epsilon)\big|\Phi(u_r, v_r)\big|\big(|\Delta u| + |\Delta v|\big), \tag{3.14}$$

where $q_1 = \max_{(u,v)\in\Omega} |(m_u, m_v)(u, v)|$, and

$$\Omega = \big\{(u, v)\colon\ u_l \leqslant u \leqslant u_r,\ z\big(u_r, v_*(u_r) - 2\delta_1\big) \leqslant z(u, v) \leqslant z\big(u_l, v_*(u_l) + 2\delta_1\big)\big\}.$$

Since

$$\big|\Phi(u_r, v_r)\big| = \big|\Phi(u_r, v_r) - \Phi\big(u_r, v_*(u_r)\big)\big| \leqslant q_2\big|v_r - v_*(u_r)\big| \leqslant q_2\delta_1,$$

where $q_2 = \max_{(u,v)\in\Omega} |\Phi_v(u,v)|$, by virtue of (3.14), we obtain the following estimate for I_1,

$$|I_1| \leqslant q_1 q_2 \delta_1 (s/\epsilon)\big(|\Delta u| + |\Delta v|\big). \tag{3.15}$$

The term I_2 is estimated as follows: first,

$$I_2 = (s/\epsilon) m(u_l, v_{\xi_1} + v_l)\big[\Phi(u_r, v_r) - \Phi(u_l, v_l)\big]$$

for some v_{ξ_1} between $\bar{v}_l - v_l$ and $\bar{v}_r - v_r$, and

$$\begin{aligned}
&\Phi(u_r, v_r) - \Phi(u_l, v_l) \\
&\quad = \big[\Phi(u_r, v_r) - \Phi\big(u_r, v_*(u_r)\big)\big] + \big[\Phi\big(u_l, v_*(u_l)\big) - \Phi(u_l, v_l)\big] \\
&\quad = \int\limits_{v_*(u_r)-v_r}^{0} \big[\Phi_v(u_r, v + v_r) - \Phi_v(u_l, v + v_l)\big]\,dv + \int\limits_{v_*(u_r)-v_r}^{v_*(u_l)-v_l} \Phi_v(u_l, v + v_l)\,dv \\
&\quad = \int\limits_{v_*(u_r)-v_r}^{0} \big[\Phi_v(u_r, v + v_r) - \Phi_v(u_l, v + v_l)\big]\,dv + \Phi_v(u_l, v_{\xi_2} + v_l)\underbrace{\big[v_*(u_l) - v_*(u_r) + \Delta v\big]}_{I_{21}}
\end{aligned}$$

for some v_{ξ_2} between $v_*(u_r) - v_r$ and $v_*(u_l) - v_l$. Then, we obtain

$$\big|I_2 - (s/\epsilon) m(u_l, v_{\xi_1} + v_l)\Phi_v(u_l, v_{\xi_2} + v_l) I_{21}\big| \leqslant q_3 q_4 \delta_1 (s/\epsilon)\big(|\Delta u| + |\Delta v|\big), \tag{3.16}$$

where $q_3 = \max_{(u,v)\in\Omega} |m(u,v)|$, and $q_4 = \max_{(u,v)\in\Omega} |(\Phi_{vu}, \Phi_{vv})(u,v)|$. A relation among Δz, Δu and Δv is needed to handle the term I_{21}, which is given by

$$\begin{aligned}
\Delta z &= z(u_r, v_r) - z(u_l, v_l) = z(u_r, v_r) - z(u_l, v_r) + z(u_l, v_r) - z(u_l, v_l) \\
&= \int\limits_{u_l}^{u_r} -a_1(u, v_r) m(u, v_r)\,du + \int\limits_{v_l}^{v_r} m(u_l, v)\,dv.
\end{aligned}$$

So, Δv can be written as

$$\Delta v = \left[\Delta z + \int\limits_{u_l}^{u_r} a_1(u, v_r) m(u, v_r)\,du\right] \Big/ m(u_l, v_{\xi_3}) \tag{3.17}$$

for some v_{ξ_3} between v_l and v_r, which also implies

$$|\Delta v| \leqslant q_5\big(|\Delta z| + |\Delta u|\big), \quad q_5 = \Big(1 + \max_{(u,v)\in\Omega}\big|(a_1 m)(u,v)\big|\Big) \Big/ \min_{(u,v)\in\Omega}\big|m(u,v)\big|. \tag{3.18}$$

We are now ready to estimate the remaining term I_{21}. It follows from (3.17) that

$$
\begin{aligned}
I_{21} &= \left[\Delta z + \int_{u_l}^{u_r} \left[a_1(u, v_r)m(u, v_r) - v_*'(u)m(u_l, v_{\xi_3})\right] du \right] \Big/ m(u_l, v_{\xi_3}) \\
&= \left\{ \Delta z + \left[a_1(u_{\xi_4}, v_r)m(u_{\xi_4}, v_r) - v_*'(u_{\xi_4})m(u_l, v_{\xi_3})\right] \Delta u \right\} / m(u_l, v_{\xi_3}) \\
&= \left\{ \Delta z + \underbrace{\left[a_1(u_{\xi_4}, v_r) - v_*'(u_{\xi_4})\right]m(u_{\xi_4}, v_r)}_{I_{23}} \Delta u \right\} / m(u_l, v_{\xi_3}) + I_{22}\Delta u
\end{aligned}
\tag{3.19}
$$

for some $u_{\xi_4} \in (u_l, u_r)$, where

$$
|I_{22}| = \left| \left[m(u_{\xi_4}, v_r) - m(u_l, v_{\xi_3})\right] v_*'(u_{\xi_4}) / m(u_l, v_{\xi_3}) \right| \leqslant q_6\big(|\Delta u| + |\Delta v|\big) \leqslant q_6 \delta_2,
$$

with $q_6 = \max_{u_l \leqslant u \leqslant u_r} |v_*'(u)| \max_{(u,v)\in\Omega} |(m_u, m_v)(u, v)| / \min_{(u,v)\in\Omega} |m(u, v)|$. By virtue of (3.16) and (3.19), we obtain

$$
\left| I_2 - (s/\epsilon)I_{24}\Delta z - (s/\epsilon)I_{24}I_{23}\Delta u \right| \leqslant q_3(q_4\delta_1 + q_2q_6\delta_2)(s/\epsilon)\big(|\Delta u| + |\Delta v|\big), \tag{3.20}
$$

where $I_{24} = m(u_l, v_{\xi_1} + v_l)\Phi_v(u_l, v_{\xi_2} + v_l)/m(u_l, v_{\xi_3})$.

It is deduced from (3.13), (3.15) and (3.20) that

$$
\left| \Delta\bar{z} - \Delta z - (s/\epsilon)I_{24}(\Delta z + I_{23}\Delta u) \right| \leqslant \underbrace{\left[q_1q_2\delta_1 + q_3(q_4\delta_1 + q_2q_6\delta_2)\right]}_{q}(s/\epsilon)\big(|\Delta u| + |\Delta v|\big),
$$

which together with (3.18) yields

$$
\Delta\bar{z} \leqslant \big(1 + (s/\epsilon)I_{24} - (s/\epsilon)qq_5\big)\Delta z + (s/\epsilon)\big(I_{23}I_{24} + q(1 + q_5)\big)\Delta u. \tag{3.21}
$$

In view of the sub-characteristic condition (1.5) namely $v_*'(u) < a_1(u, v_r)$ for all $u \in [u_l, u_r]$, the positivity of function m and (1.3), we have

$$
I_{23} > 0, \qquad I_{24} < 0.
$$

Therefore, we can choose δ_1, δ_2 and δ_3 so small that

$$
I_{23}I_{24} + q(1 + q_5) \leqslant 0 \quad \text{and} \quad 1 + (s/\epsilon)I_{24} - (s/\epsilon)qq_5 \geqslant 0,
$$

which together with (3.21) gives $\Delta\bar{z} \leqslant 0$.

Step 2. In this step, we prove that $\bar{v}_r \geqslant s_2(u_r; (u_l, \bar{v}_l))$. This is proved by considering two different cases.

Case 1. $z(u_r, \bar{v}_r) - z(u_r, s_2(u_r; (u_l, \bar{v}_l))) \geqslant z(u_r, v_r) - z(u_r, s_2(u_r; (u_l, v_l)))$.

In view of (3.12), we see that $v_r \geqslant s_2(u_r; (u_l, v_l))$, which implies that

$$
z(u_r, v_r) - z\big(u_r, s_2\big(u_r; (u_l, v_l)\big)\big) = \int_{s_2(u_r;(u_l,v_l))}^{v_r} m(u_r, v)\, dv \geqslant 0.
$$

So, in this case, we have

$$z(u_r, \bar{v}_r) - z\big(u_r, s_2\big(u_r; (u_l, \bar{v}_l)\big)\big) \geqslant 0,$$

i.e., $\bar{v}_r \geqslant s_2(u_r; (u_l, \bar{v}_l))$.

Case 2. $z(u_r, \bar{v}_r) - z(u_r, s_2(u_r; (u_l, \bar{v}_l))) < z(u_r, v_r) - z(u_r, s_2(u_r; (u_l, v_l)))$.

In this case, we have

$$\begin{aligned} 0 &> \big[z(u_r, \bar{v}_r) - z(u_r, v_r)\big] - \big[z\big(u_r, s_2\big(u_r; (u_l, \bar{v}_l)\big)\big) - z\big(u_r, s_2\big(u_r; (u_l, v_l)\big)\big)\big] \\ &= I_1 + I_2 - \big[z\big(u_r, s_2\big(u_r; (u_l, \bar{v}_l)\big)\big) - z(u_l, \bar{v}_l)\big] + \big[z\big(u_r, s_2\big(u_r; (u_l, v_l)\big)\big) - z(u_l, v_l)\big], \end{aligned}$$

where I_1 and I_2 are given by (3.13). Since

$$z\big(u_r, s_2\big(u_r; (u_l, \bar{v}_l)\big)\big) - z(u_l, \bar{v}_l) = \int_{u_l}^{u_r} (mh_2 - ma_1)\big(u, s_2\big(u; (u_l, \bar{v}_l)\big)\big)\, du$$

and

$$z\big(u_r, s_2\big(u_r; (u_l, v_l)\big)\big) - z(u_l, v_l) = \int_{u_l}^{u_r} (mh_2 - ma_1)\big(u, s_2\big(u; (u_l, v_l)\big)\big)\, du,$$

where $h_2(u, s_2(u; P_0)) = ds_2(u; P_0)/du$, we have

$$I_1 + I_2 < \int_{u_l}^{u_r} \big[\big(m(h_2 - a_1)\big)\big(u, s_2\big(u; (u_l, \bar{v}_l)\big)\big) - \big(m(h_2 - a_1)\big)\big(u, s_2\big(u; (u_l, v_l)\big)\big)\big]\, du,$$

or alternatively,

$$|I_1 + I_2| < q_7|\Delta u||\bar{v}_l - v_l| = q_7(s/\epsilon)\big|\Phi(u_l, v_l)\big||\Delta u| \leqslant q_2 q_7 \delta_1 (s/\epsilon)|\Delta u|, \tag{3.22}$$

where

$$q_7 = \max_{(u,v)\in\Omega} \big|\big(m(h_2 - a_1)\big)_v(u, v)\big| \max_{(u,v)\in\Omega} \big|\partial_v s_2\big(u; (u_l, v)\big)\big|.$$

By use of (3.15), (3.18), (3.20) and (3.22), we obtain

$$(s/\epsilon) I_{24} \Delta z + (s/\epsilon) I_{24} I_{23} \Delta u < (q + q_2 q_7 \delta_1)(s/\epsilon)\big[(1 + q_5)|\Delta u| + q_5|\Delta z|\big]$$

namely

$$\big[I_{24} + (q + q_2 q_7 \delta_1) q_5\big]\Delta z + \big[I_{24} I_{23} - (q + q_2 q_7 \delta_1)(1 + q_5)\big]\, \Delta u < 0,$$

which implies that

$$\begin{aligned}\Delta z &> -\Delta u\big[I_{24}I_{23} - (q+q_2q_7\delta_1)(1+q_5)\big]/\big[I_{24} + (q+q_2q_7\delta_1)q_5\big] \\ &= -I_{23}\Delta u + \Delta u\,\underbrace{(q+q_2q_7\delta_1)(q_5I_{23}+1+q_5)/\big[I_{24}+(q+q_2q_7\delta_1)q_5\big]}_{I_3}.\end{aligned} \tag{3.23}$$

Here we require δ_1 and δ_2 are small such that $I_{24} + (q+q_2q_7\delta_1)q_5 < 0$. Since $\Delta z \leqslant 0$ due to (3.12), we also have, with the help of (3.23),

$$|\Delta z| \leqslant I_{23}\Delta u + |I_3|\Delta u. \tag{3.24}$$

By the same way as in Step 1, we can set $\Delta\omega = \omega(u_r, v_r) - \omega(u_l, v_l)$ and write

$$\Delta\bar{\omega} \equiv \omega(u_r, \bar{v}_r) - \omega(u_l, \bar{v}_l) = \Delta\omega + \tilde{I}_1 + \tilde{I}_2,$$

where

$$\tilde{I}_1 = \int_0^{\bar{v}_r - v_r} \big[n(u_r, v+v_r) - n(u_l, v+v_l)\big]\,dv, \qquad \tilde{I}_2 = \int_{\bar{v}_l - v_l}^{\bar{v}_r - v_r} n(u_l, v+v_l)\,dv.$$

Then, by (3.18) and (3.24), we have

$$|\Delta\bar{\omega} - \Delta\omega| \leqslant q_8(s/\epsilon)\big(|\Delta u| + |\Delta v|\big) \leqslant q_8(s/\epsilon)\big(1 + q_5 + q_5\big(I_{23} + |I_3|\big)\big)|\Delta u|, \tag{3.25}$$

where

$$q_8 = q_2\delta_1 \max_{(u,v)\in\Omega}\big|(n_u, n_v)(u,v)\big| + \max_{(u,v)\in\Omega}\big|(\Phi_u, \Phi_v)(u,v)\big| \max_{(u,v)\in\Omega}\big|n(u,v)\big|.$$

The term $\Delta\omega$ can be rewritten as

$$\begin{aligned}\Delta\omega &= \int_{u_l}^{u_r} -a_2(u, v_r)n(u, v_r)\,du + \int_{v_l}^{v_r} n(u_l, v)\,dv \\ &= \int_{u_l}^{u_r} -a_2(u, v_r)n(u, v_r)\,du + n(u_l, v_{\xi_5})\Delta v\end{aligned}$$

for some v_{ξ_5} between v_l and v_r. Since

$$\begin{aligned}\Delta v &> \left[\int_{u_l}^{u_r} a_1(u, v_r)m(u, v_r)\,du - I_{23}\Delta u + I_3\Delta u\right]\Big/ m(u_l, v_{\xi_3}) \\ &= \frac{1}{m(u_l, v_{\xi_3})}\int_{u_l}^{u_r} v_*'(u)m(u, v_r)\,du + \underbrace{\frac{[a_1(u_\xi, v_r) - v_*'(u_\xi)]m(u_\xi, v_r) - I_{23} + I_3}{m(u_l, v_{\xi_3})}}_{I_4}\Delta u\end{aligned}$$

for some $u_\xi \in (u_l, u_r)$ because of (3.17) and (3.23), then we obtain that

$$\Delta\omega > \int_{u_l}^{u_r} -a_2(u, v_r)n(u, v_r)\,du + n(u_l, v_{\xi_5})\left[\int_{u_l}^{u_r} v'_*(u)\frac{m(u, v_r)}{m(u_l, v_{\xi_3})}\,du + I_4\Delta u\right],$$

which yields that

$$\begin{aligned}
\Delta\omega &> n(u_l, v_{\xi_5})\int_{u_l}^{u_r}\left[\frac{m(u, v_r)}{m(u_l, v_{\xi_3})}v'_*(u) - \frac{n(u, v_r)}{n(u_l, v_{\xi_5})}a_2(u, v_r)\right]du + n(u_l, v_{\xi_5})I_4\Delta u \\
&= n(u_l, v_{\xi_5})\int_{u_l}^{u_r}\left[v'_*(u) - a_2(u, v_r)\right]du + n(u_l, v_{\xi_5})(I_4\Delta u + I_5) \\
&= n(u_l, v_{\xi_5})\left[v'_*(u_{\xi_6}) - a_2(u_{\xi_6}, v_r)\right]\Delta u + n(u_l, v_{\xi_5})(I_4\Delta u + I_5), \qquad (3.26)
\end{aligned}$$

for some $u_{\xi_6} \in (u_l, u_r)$. Here the terms I_4 and I_5 can be estimated as follows:

$$|I_4| \leqslant q_9\delta_2 + |I_3|/m(u_l, v_{\xi_3})$$

and

$$|I_5| = \left|\int_{u_l}^{u_r}\left[\left(\frac{m(u, v_r)}{m(u_l, v_{\xi_3})} - 1\right)v'_*(u) - \left(\frac{n(u, v_r)}{n(u_l, v_{\xi_5})} - 1\right)a_2(u, v_r)\right]du\right| \leqslant q_{10}\delta_2\Delta u,$$

where

$$q_9 = \max_{(u,v)\in\Omega}\left|\left(\left((a_1 - v'_*)m\right)_u, \left((a_1 - v'_*)m\right)_v\right)(u, v)\right| / \min_{(u,v)\in\Omega} m(u, v)$$

and

$$q_{10} = \frac{\max_{(u,v)\in\Omega}|(m_u, m_v)(u, v)|}{\min_{(u,v)\in\Omega} m(u, v)}\max_{u_l\leqslant u\leqslant u_r}\left|v'_*(u)\right| + \frac{\max_{(u,v)\in\Omega}|(n_u, n_v)(u, v)|}{\min_{(u,v)\in\Omega} n(u, v)}\max_{u_l\leqslant u\leqslant u_r}\left|a_2(u, v_r)\right|.$$

It follows from (3.25) and (3.26) that

$$\begin{aligned}
\Delta\bar{\omega} &> n(u_l, v_{\xi_5})\left[v'_*(u_{\xi_6}) - a_2(u_{\xi_6}, v_r)\right]\Delta u - q_{11}\left((q_9 + q_{10})\delta_2 + |I_3|/q_{12}\right)\Delta u \\
&\quad - (s/\epsilon)q_8\left(1 + q_5 + q_5(I_{23} + |I_3|)\right)|\Delta u|,
\end{aligned}$$

where $q_{11} = \max_{(u,v)\in\Omega} n(u, v)$ and $q_{12} = \min_{(u,v)\in\Omega} m(u, v)$. In view of the sub-characteristic condition (1.5) namely $a_2(u, v_r) < v'_*(u)$, we can choose δ_1, δ_2 and s/ϵ so sufficiently small that $\Delta\bar{\omega} \geqslant 0$, which implies $\bar{v}_r \geqslant s_2(u_r; (u_l, \bar{v}_l))$. This completes the proof of the lemma.

3.2. Proof of Theorem 3.1

Theorem 3.1 can be proved by using Lemmas 2.2, 3.2 and the properties for the Riemann solutions for the homogeneous system (3.3).

Step 1. First, we give some properties of the solution set B. In view of the sub-characteristic condition (1.5) namely $a_1(u,v) > v_*'(u)$ for all $(u,v) \in B$, we have

$$w_1\big(u;\big(u^-, v_*\big(u^-\big)\big)\big) - v_*(u) = \int_{u^-}^{u} \big[a_1\big(u, w_1\big(u;\big(u^-, v_*\big(u^-\big)\big)\big)\big) - v_*'(u)\big] \geqslant 0, \quad u^- \leqslant u \leqslant u^+.$$

Similarly

$$w_1\big(u;\big(u^+, v_*\big(u^+\big)\big)\big) - v_*(u) \leqslant 0, \quad u^- \leqslant u \leqslant u^+,$$

which implies that

$$w_1\big(u;\big(u^+, \min\big(v^+, v_*\big(u^+\big)\big)\big)\big) \leqslant v_*(u) \leqslant w_1\big(u;\big(u^-, \max\big(v^-, v_*\big(u^-\big)\big)\big)\big), \quad u^- \leqslant u \leqslant u^+. \tag{3.27}$$

So, for any $(u_i, v_i) \in B$, $i = 1, 2$, we have

$$\begin{aligned} |v_1 - v_2| &\leqslant \max_{u^- \leqslant u \leqslant u^+} \big|w_1\big(u;\big(u^-, \max\big(v^-, v_*\big(u^-\big)\big)\big)\big) - w_1\big(u;\big(u^+, \min\big(v^+, v_*\big(u^+\big)\big)\big)\big)\big| \\ &\leqslant 2 \max_{(u,v)\in B} \big|a_1(u,v)\big|\big(u^+ - u^-\big) + \big|\max\big(v^-, v_*\big(u^-\big)\big) - \min\big(v^+, v_*\big(u^+\big)\big)\big| \\ &\leqslant \max_{(u,v)\in B} \big(2\big|a_1(u,v)\big| + \big|v_*'(u)\big|\big)\big(u^+ - u^-\big) + \big|v^- - v_*\big(u^-\big)\big| + \big|v^+ - v_*\big(u^+\big)\big| \\ &\leqslant 2 \max_{(u,v)\in B} \big(\big|a_1(u,v)\big| + \big|v_*'(u)\big|\big)\big(u^+ - u^-\big) + 2\big|v^- - v_*\big(u^-\big)\big| + \big|v^+ - v^-\big|. \end{aligned} \tag{3.28}$$

It follows from (A2) and Lemma 2.2 that (u^-, v^-) and (u^+, v^+) are connected by a 1-rarefaction wave and a 2-shock wave. Then there exists a state $(\hat{u}, \hat{v})$ such that (u^-, v^-) and $(\hat{u}, \hat{v})$ are connected by 1-rarefaction wave, $(\hat{u}, \hat{v})$ and (u^+, v^+) are connected by S_2. That is:

$$u^- \leqslant \hat{u} \leqslant u^+, \qquad \hat{v} = w_1\big(\hat{u};\big(u^-, v^-\big)\big), \qquad v^+ = s_2\big(u^+;(\hat{u}, \hat{v})\big).$$

This gives

$$\big|\hat{v} - v^-\big| = \left|\int_{u^-}^{\hat{u}} a_1\big(u, w_1\big(\hat{u};\big(u^-, v^-\big)\big)\big)\,du\right| \leqslant \max_{(u,v)\in B} \big|a_1(u,v)\big|\big(\hat{u} - u^-\big), \tag{3.29}$$

and

$$\big|v^+ - \hat{v}\big| = \left|\int_{\hat{u}}^{u^+} \frac{d}{du} s_2\big(u;(\hat{u}, \hat{v})\big)\,du\right| \leqslant \max_{(\bar{u},\bar{v})\in B,\, u^- \leqslant u \leqslant u^+} \left|\frac{d}{du} s_2\big(u;(\bar{u}, \bar{v})\big)\right|\big(u^+ - \hat{u}\big). \tag{3.30}$$

It yields from (3.29) and (3.30) that

$$\left|v^+ - v^-\right| \leqslant k_1\left(u^+ - u^-\right), \tag{3.31}$$

where

$$k_1 = \max\left\{\max_{(u,v)\in B}\left|a_1(u,v)\right|,\ \max_{(\bar{u},\bar{v})\in B,\, u^-\leqslant u\leqslant u^+}\left|\frac{d}{du}s_2\left(u;(\bar{u},\bar{v})\right)\right|\right\}. \tag{3.32}$$

In view of (3.28) and (3.31), we have

$$|u_1 - u_2| \leqslant u^+ - u^-, \qquad |v_1 - v_2| \leqslant 5k_2\left(u^+ - u^-\right) + 2\left|v^- - v_*\left(u^-\right)\right|,$$

for any $(u_i, v_i) \in B$, $i = 1, 2$, where

$$k_2 = \max\left\{k_1,\ \max_{u^-\leqslant u\leqslant u^+}\left|v_*'(u)\right|\right\}.$$

Since $(u, v_*(u)) \in B$ for all $u \in [u^-, u^+]$ thanks to (3.27), we also get

$$\left|v - v_*(u)\right| \leqslant 5k_2\left(u^+ - u^-\right) + 2\left|v^- - v_*\left(u^-\right)\right| \tag{3.33}$$

for any $(u, v) \in B$. Therefore, the conditions in Lemma 3.2 are satisfied for any $(u_l, v_l) \in B$ and $(u_r, v_r) \in B$, if $|u^+ - u^-| + |v^- - v_*(u^-)|$ is small.

Step 2. Let us go back to the scheme for time step T_0. It follows from (A2) that

$$\left(u_0(x), v_0(x)\right) \in B \quad \text{for any } x \in \mathbb{R},$$

which together with Lemma 3.2 gives

$$\left(u_0(x_2), v_0(x_2) + (s/\epsilon)\Phi(u_0, v_0)(x_2)\right) \in C_2\left(u_0(x_1), v_0(x_1) + (s/\epsilon)\Phi(u_0, v_0)(x_1)\right) \tag{3.34}$$

for any $x_1 < x_2$. Noting that for any $(u, v) \in B$,

$$\begin{aligned}\left[v + (s/\epsilon)\Phi(u,v)\right] - v_*(u) &= \left[v - v_*(u)\right] + (s/\epsilon)\left[\Phi(u,v) - \Phi\left(u, v_*(u)\right)\right] \\ &= \left[v - v_*(u)\right]\left[1 + (s/\epsilon)\Phi_v(u,\xi)\right]\end{aligned}$$

for some ξ between v and $v_*(u)$, we have, by (3.1), that

$$\left|\left[v + (s/\epsilon)\Phi(u,v)\right] - v_*(u)\right| \leqslant \left|v - v_*(u)\right|,$$

namely

$$\left(u, v + (s/\epsilon)\Phi(u,v)\right) \in B, \quad \text{for any } (u,v) \in B. \tag{3.35}$$

It follows from (3.34) and (3.35) that

$$\left(u^s(x,0+), v^s(x,0+)\right) = \left(u^s(x,0-), v^s(x,0-) + (s/\epsilon)\Phi\left(u^s, v^s\right)(x,0-)\right) \in B, \tag{3.36}$$

for any $x \in \mathbb{R}$. Hence, we obtain $(u^s, v^s)(x,t)$ in T_0 by solving the homogeneous problem (3.3) (the reader may refer to [20] for details). Moreover, $(u^s, v^s)(x,t)$ satisfies the following properties: for any $x_1 < x_2$ and $0 \leqslant t < s$

$$u^- \leqslant u^s(x_1,t) \leqslant u^s(x_2,t) \leqslant u^+, \qquad \big(u^s, v^s\big)(x_2,t) \in C_2\big(\big(u^s, v^s\big)(x_1,t)\big), \tag{3.37}$$

and

$$TV\, v^s(\cdot,t) \leqslant k_1\big(u^+ - u^-\big),$$

where k_1 is given by (3.32). By virtue of (3.36) and (3.37), we obtain

$$\big(u^s, v^s\big)(x,t) \in B_0 \subseteq B,$$

for any $x \in \mathbb{R}$ and $t \in (0,s)$, where

$$B_0 = \big\{(u,v)\colon\ u^- \leqslant u \leqslant u^+,\ z\big(u^+, v^s(\infty, 0+)\big) \leqslant z(u,v) \leqslant z\big(u^-, v^s(-\infty, 0+)\big)\big\}.$$

The same procedure as for T_0 gives the solution on the time steps $T_1, \ldots, T_n, \ldots,$ which satisfies (3.4)–(3.6).

Step 3. Proof of (3.7). Suppose $t_2 > t_1$, and

$$t_0 = \sup\{t \leqslant t_1\colon\ t = ns \text{ for some } n\}.$$

Let $J = [(t_2 - t_0)/s] + 1$, where $[(t_2 - t_0)/s]$ is the integer part of $(t_2 - t_0)/s$, then

$$Js \leqslant t_2 - t_1 + 2s. \tag{3.38}$$

Now $(u^s, v^s)(x, t_i)$ $(i = 1, 2)$ are determined by the Cauchy data

$$\big\{\big(u^s, v^s\big)(y, t_0)\colon\ y \in I\big\},$$

where $I = [x - Jr, x + Jr]$. It follows from the proof of (3.6) and the property (3.4) that

$$\big|u^s(x,t_2) - u^s(x,t_1)\big| \leqslant u^s(x + Jr, t_0) - u^s(x - Jr, t_0) = T.V.\big\{u^s(y,t_0)\colon\ y \in I\big\}.$$

Hence, for integer m in the scheme,

$$\int_{mr}^{(m+2)r} \big|u^s(x,t_2) - u^s(x,t_1)\big|\,dx \leqslant 2rT.V.\big\{u^s(y,t_0)\colon\ y \in I_m\big\},$$

where $I_m = [(m-J)r, (m+2+J)r]$. Therefore,

$$\int_{-\infty}^{\infty} \left|u^s(x,t_2) - u^s(x,t_1)\right| dx = \frac{1}{2} \sum_m \int_{mr}^{(m+2)r} \left|u^s(x,t_2) - u^s(x,t_1)\right| dx$$
$$\leqslant r \sum_m T.V.\left\{u^s(y,t_0)\colon\ y \in I_m\right\}$$
$$\leqslant 2r \sum_m T.V.\left\{u^s(y,t_0)\colon\ y \in I'_m\right\},$$

where $I'_m = [mr, (m+2+J)r]$. It then follows, by (3.4) and (3.38), that

$$\begin{aligned}\int_{-\infty}^{\infty} \left|u^s(x,t_2) - u^s(x,t_1)\right| dx &\leqslant 2r(J+1)T.V.u^s(\cdot,t_0) \\ &= 2r(J+1)\left(u^+ - u^-\right) = 2(r/s)(J+1)s\left(u^+ - u^-\right) \\ &\leqslant C\left[(t_2 - t_1) + s\right]\left(u^+ - u^-\right), \end{aligned} \tag{3.39}$$

where C is a constant independent of ϵ, s, r and θ. This proves (3.7).

Step 4. Proof of (3.8). Note that

$$\begin{aligned}\left|v^s(x,t_2) - v^s(x,t_1)\right| &= \left|v^s(x,t_2) - v^s\big(x,(n+J-1)s\big)\right| \\ &\quad + \left|v^s\big(x,(n+J-1)s\big) - v^s\big(x,(n+J-2)s\big)\right| \\ &\quad + \left|v^s\big(x,(n+J-2)s\big) - v^s\big(x,(n+J-3)s\big)\right| \\ &\quad + \cdots + \left|v^s\big(x,(n+1)s\big) - v^s(x,ns)\right| + \left|v^s(x,t_0) - v^s(x,t_1)\right| \\ &\equiv I_0 + I_1 + I_2 + \cdots + I_{J-1} + I_J. \end{aligned} \tag{3.40}$$

For any $x \in \mathbb{R}$, there exists an integer m_0 such that $x \in [m_0 r, (m_0+1)r)$. We consider only the case $m_0 + n + J = even$, since it is the same in the case $m_0 + n + J = odd$. For I_0, it follows from the construction of (u^s, v^s) that $(u^s(x,t_2), v^s(x,t_2)) \in C_2((u^s(x,(n+J-1)s), v^s(x,(n+J-1)s)))$; and then, by (3.31), that

$$I_0 \leqslant k_1\left[u^s(x,t_2) - u^s\big(x,(n+J-1)s\big)\right]. \tag{3.41}$$

For I_1, since there exists a constant state (u_α, v_α) such that

$$u^s\big(x,(n+J-1)s\big) = u_\alpha, \qquad v^s\big(x,(n+J-1)s\big) = v_\alpha + (s/\epsilon)\Phi(u_\alpha, v_\alpha)$$

and $(u^s(x,(n+J-2)s), v^s(x,(n+J-2)s)) \in C_2((u_\alpha, v_\alpha))$, we obtain, in view of (3.31), that

$$\begin{aligned} I_1 &= \left|v_\alpha + (s/\epsilon)\Phi(u_\alpha, v_\alpha) - v^s\big(x,(n+J-2)s\big)\right| \\ &\leqslant \left|v_\alpha - v^s\big(x,(n+J-2)s\big)\right| + (s/\epsilon)\left|\Phi(u_\alpha, v_\alpha)\right| \\ &\leqslant k_1\left[u^s\big(x,(n+J-2)s\big) - u^s\big(x,(n+J-1)s\big)\right] \\ &\quad + (s/\epsilon) \max_{(u,v)\in B}\left|\Phi_v(u,v)\right|\left|v_\alpha - v_*(u_\alpha)\right|. \end{aligned} \tag{3.42}$$

For I_2, similarly as I_1, we have

$$I_2 \leqslant k_1\big[u^s\big(x,(n+J-2)s\big) - u^s\big(x,(n+J-3)s\big)\big] + (s/\epsilon)\max_{(u,v)\in B}\big|\Phi_v(u,v)\big|\big|v_\beta - v_*(u_\beta)\big|, \tag{3.43}$$

where (u_β, v_β) satisfies $(u_\beta, v_\beta) \in C_2((u^s(x,(n+J-3)s), v^s(x,(n+J-3)s)))$ and

$$u^s\big(x,(n+J-2)s\big) = u_\beta, \qquad v^s\big(x,(n+J-2)s\big) = v_\beta + (s/\epsilon)\Phi(u_\beta, v_\beta).$$

The estimates for $I_3, \dots, I_{J-1}$ are the same as that for I_1. By the same estimate for I_0, one has

$$I_J \leqslant k_1\big[u^s(x,t_1) - u^s(x,t_0)\big]. \tag{3.44}$$

It yields from (3.40)–(3.44) that

$$\begin{aligned}
&\big|v^s(x,t_2) - v^s(x,t_1)\big| \\
&\quad\leqslant k_1\big\{\big[u^s(x,t_2) - u^s\big(x,(n+J-1)s\big)\big] + \big[u^s(x,t_1) - u^s(x,t_0)\big]\big\} \\
&\qquad + (s/\epsilon)\max_{(u,v)\in B}\big|\Phi_v(u,v)\big|\big[\big|v_\alpha - v_*(u_\alpha)\big| + \big|v_\beta - v_*(u_\beta)\big| + \cdots + \big|v_\gamma - v_*(u_\gamma)\big|\big],
\end{aligned} \tag{3.45}$$

where $u^s(x,(n+1)s) = u_\gamma$ and $v^s(x,(n+1)s) = v_\gamma + (s/\epsilon)\Phi(u_\gamma, v_\gamma)$ (or equivalently $(u_\gamma, v_\gamma) = (u^s, v^s)(x_\gamma, t_\gamma)$ for some pairs $(x_\gamma, t_\gamma) \in (x-r, x+r) \times (t_0, (n+1)s)$). It remains to estimate the sum of $|v_\alpha - v_*(u_\alpha)|$, $|v_\beta - v_*(u_\beta)|$, and so on. Note that

$$\begin{aligned}
&\big|v_\alpha - v_*(u_\alpha)\big| \\
&\quad\leqslant \big|\big[v_\alpha - v_*(u_\alpha)\big] - \big[v^s - v_*\big(u^s\big)\big]\big(x,(n+J-2)s\big)\big| + \big|\big[v^s - v_*\big(u^s\big)\big]\big(x,(n+J-2)s\big)\big| \\
&\quad\leqslant \big|v_\alpha - v^s\big(x,(n+J-2)s\big)\big| + \max_{u^-\leqslant u\leqslant u^+}\big|v_*'(u)\big|\big|u_\alpha - u^s\big(x,(n+J-2)s\big)\big| \\
&\qquad + \big|v_\beta + (s/\epsilon)\Phi(u_\beta, v_\beta) - v_*(u_\beta)\big| \\
&\quad\leqslant k^*\big(u^s\big(x,(n+J-2)s\big) - u^s\big(x,(n+J-1)s\big)\big) + \phi^*\big|v_\beta - v_*(u_\beta)\big|,
\end{aligned} \tag{3.46}$$

where

$$k^* = k_1 + \max_{u^-\leqslant u\leqslant u^+}\big|v_*'(u)\big| \quad \text{and} \quad \phi^* = 1 - (s/\epsilon)\min_{(u,v)\in B}\big|\Phi_v(u,v)\big| > 0$$

due to (3.1); we can use (3.7) to obtain that

$$\begin{aligned}
&\int_{-L}^{L}\big[\big|v_\alpha - v_*(u_\alpha)\big| + \big|v_\beta - v_*(u_\beta)\big| + \cdots + \big|v_\gamma - v_*(u_\gamma)\big|\big]\,dx \\
&\quad\leqslant Cs\big[(J-2) + (J-3)\phi^* + \cdots + \big(\phi^*\big)^{J-3}\big] \\
&\qquad + \big(1 + \phi^* + \cdots + \big(\phi^*\big)^{J-2}\big)\int_{-L}^{L}\big|v_\gamma - v_*(u_\gamma)\big|\,dx,
\end{aligned}$$

where C is a constant independent of ϵ, s, r, θ and L. It hence follows from (3.45) that

$$\begin{aligned}\int_{-L}^{L} \left|v^s(x,t_2) - v^s(x,t_1)\right| dx &\leqslant Cs + C(s/\epsilon)s\left[(J-2) + (J-3)\phi^* + \cdots + \left(\phi^*\right)^{J-3}\right] \\ &\quad + C(s/\epsilon)\left(1 - \left(\phi^*\right)^{J-1}\right)/\left(1-\phi^*\right) \int_{-L}^{L} \left|v_\gamma - v_*(u_\gamma)\right| dx \\ &\leqslant Cs + Cs(J-2) + C\left(1 - \left(\phi^*\right)^{J-1}\right) \int_{-L}^{L} \left|v_\gamma - v_*(u_\gamma)\right| dx \\ &\leqslant C(s + t_2 - t_1)\left(1 + \epsilon^{-1} \int_{-L}^{L} \left|v_\gamma - v_*(u_\gamma)\right| dx\right), \end{aligned} \tag{3.47}$$

where C is a constant independent of ϵ, s, r, θ and L. Here we have used the fact, by Taylor's expansion, that

$$\begin{aligned}\left|1 - \left(\phi^*\right)^{J-1}\right| &= \left|1 - \left[1 - (s/\epsilon) \min_{(u,v)\in B}\left|\Phi_v(u,v)\right|\right]^{J-1}\right| = \left|1 - \vartheta^{(s\bar{c}/\epsilon)(J-1)}\right| \\ &= \left|(s\bar{c}/\epsilon)(J-1)\vartheta^{\bar{\xi}(s\bar{c}/\epsilon)(J-1)} \ln\vartheta\right| \leqslant C\epsilon^{-1}(t_2 - t_1 + s),\end{aligned}$$

where $\bar{c} = \min_{(u,v)\in B} |\Phi_v(u,v)|$, $\vartheta = [1 - (s/\epsilon)\bar{c}]^{\epsilon/(s\bar{c})} \in (0,1)$ thanks to the smallness of s/ϵ, the constant $\bar{\xi} \in (0,1)$, and the constant C is independent of ϵ, s, r, θ and L. Therefore, we prove (3.8), using (3.47) and (3.33).

Step 5. Proof of (3.9). As the notations in the proof of (3.8), we may choose $t_2 = t$ and $t_1 = 0$. Since $(u^s, v^s)(x,t) \in C_2((u^s, v^s)(x,(J-1)s))$, in the same way as in (3.46), we have

$$\begin{aligned}&\left|v^s(x,t) - v_*\left(u^s(x,t)\right)\right| \\ &\quad = \left|\left[v^s(x,t) - v_*\left(u^s(x,t)\right)\right] - \left[v^s\left(x,(J-1)s\right) - v_*\left(u^s\left(x,(J-1)s\right)\right)\right]\right| \\ &\qquad + \left|v^s\left(x,(J-1)s\right) - v_*\left(u^s\left(x,(J-1)s\right)\right)\right| \\ &\quad \leqslant k^*\left(u^s(x,t) - u^s\left(x,(J-1)s\right)\right) + \phi^*\left|v_a - v_*(u_a)\right|,\end{aligned} \tag{3.48}$$

where $u^s(x,(J-1)s) = u_a$ and $v^s(x,(J-1)s) = v_a + (s/\epsilon)\Phi(u_a,v_a)$. Since $(u^s,v^s)(x,(J-2)s) \in C_2((u_a,v_a))$, then it gives similarly that

$$\begin{aligned}&\left|v_a - v_*(u_a)\right| \\ &\quad \leqslant k^*\left(u^s\left(x,(J-2)s\right) - u_a\right) + \phi^*\left|v_b - v_*(u_b)\right| \\ &\quad = k^*\left(u^s\left(x,(J-2)s\right) - u^s\left(x,(J-1)s\right)\right) + \phi^*\left|v_b - v_*(u_b)\right|,\end{aligned} \tag{3.49}$$

where $u^s(x,(J-2)s) = u_b$ and $v^s(x,(J-2)s) = v_b + (s/\epsilon)\Phi(u_b,v_b)$. It follows from (3.48) and (3.49) that

$$\int_{-L}^{L} \left|v^s(x,t) - v_*\big(u^s(x,t)\big)\right| dx$$
$$\leqslant k^* \int \big(u^s(x,t) - u^s\big(x,(J-1)s\big)\big)\,dx + k^*\phi^* \int \big(u^s\big(x,(J-2)s\big) - u^s\big(x,(J-1)s\big)\big)\,dx$$
$$+ \cdots + k^*\big(\phi^*\big)^{J-1} \int \left|u^s(x,s) - u^s(x,0)\right| dx + \big(\phi^*\big)^J \int_{-L}^{L} \left|v_c - v_*(u_c)\right| dx,$$

where $(u_c, v_c) = (u_0, v_0)(x_c)$ for a certain $x_c \in (-\infty, \infty)$; and then, by (3.33) and (3.7), that

$$\int_{-L}^{L} \left|v^s(x,t) - v_*\big(u^s(x,t)\big)\right| dx \leqslant C\left[\frac{2s}{1-\phi^*} + 2L\big(\phi^*\big)^J\right] \leqslant C\big[\epsilon + 2L\vartheta^{\bar{c}t/\epsilon}\big],$$

where C is a constant independent of ϵ, s, r, θ and L. Here we have used the estimate

$$\big(\phi^*\big)^J = \Big[1 - (s/\epsilon)\min_{(u,v)\in B}\left|\Phi_v(u,v)\right|\Big]^{[t/s]+1} = \vartheta^{(([t/s]+1)s\bar{c}/\epsilon)} \leqslant \vartheta^{\bar{c}t/\epsilon},$$

where $\bar{c} = \min_{(u,v)\in B}|\Phi_v(u,v)|$ and $\vartheta = [1 - (s/\epsilon)\bar{c}]^{\epsilon/(s\bar{c})}$ is strictly less than 1. This proves (3.9) and completes the proof of Theorem 3.1.

3.3. Proofs of Theorems 2.4 and 2.5

With Theorem 3.1, the proof of Theorem 2.4 follows from the same argument as in [5], except the estimate (2.13). We now prove (2.13), using the same notations as in the proof of (3.7). It follows from (3.47) that

$$\int_{-L}^{L} \left|v^s(x,t_2) - v^s(x,t_1)\right| dx \leqslant C(s + t_2 - t_1)\left(1 + \epsilon^{-1}\int_{-L}^{L} \left|v_\gamma - v_*(u_\gamma)\right| dx\right), \tag{3.50}$$

where $(u_\gamma, v_\gamma) = (u^s, v^s)(x_\gamma, \bar{t})$ for the fixed time $\bar{t} \in (t_0, (n+1)s)$ and the corresponding $x_\gamma \in (x-r, x+r)$, and C is a constant independent of ϵ, s, r, θ and L. Since for any $x, y \in (mr, (m+2)r)$ when $m+n = even$, we have $x_\gamma = y_\gamma$, denoted by $x_m \in (mr, (m+2)r)$; then

$$\int_{mr}^{(m+2)r} \left|v_\gamma - v_*(u_\gamma)\right| dx = 2r\left|v^s(x_m,\bar{t}) - v_*\big(u^s(x_m,\bar{t})\big)\right|,$$

which means when $r \to 0$,

$$\int_{-L}^{L} \left|v_\gamma - v_*(u_\gamma)\right| dx = \int_{-L}^{L} \left|v^s(x,\bar{t}) - v_*\big(u^s(x,\bar{t})\big)\right| dx \leqslant C\big[\epsilon + L\nu^{c\bar{t}/\epsilon}\big]$$

where (3.9) is used in the last inequality. So, we have, from (3.50), that

$$\int_{-L}^{L} \left| v^{\epsilon}(x,t_2) - v^{\epsilon}(x,t_1) \right| dx \leqslant C(t_2 - t_1)\left(1 + L\epsilon^{-1}\nu^{ch/\epsilon}\right),$$

for $t_2 > t_1 \geqslant h > 0$, where constants $C, c, \nu \in (0,1)$ are independent of ϵ, s, r, θ and L. This proves (2.13).

With Theorem 2.4, the proof of Theorem 2.5 follows from the same arguments as in [15,17].

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